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SOME MECHANICAL DEVICES TO GENERATE CERTAIN SYSTEMS OF CURVES.

By ARNOLD EMCH, University of Illinois.

1. The construction of curves and surfaces by mechanical means, especially by linkages, has been the object of a number of investigations. Some of the most important results along these lines are due to Kempe* and Koenigs.† The first proved that all algebraic curves in a plane may be generated by link-motions. Koenigs found the same to be true for algebraic curves in space. In a further generalization the writer has shown‡ that every algebraic transformation between any number of variables, and as a consequence between any number of complex variables, may be realized by linkages. In another place§ I have, in particular, described linkages for continuous groups of collineation in a plane. These imply the description of conics.

Based upon an articulated regulus|| it is possible to perform by link-motions certain transformations of a plane into particular quadrics and quartic surfaces and to describe twisted curves of higher order.¶

There are, however, a great number of other mechanical and physical devices besides link-motions by which various systems of curves, for example Lissajous curves in acoustics, may be generated. The following simple examples, in which again ruled quadrics appear, may be of interest in this respect.

2. Consider a variable string model of a regulus in which strings of equal length are used. One end of each string is attached to a point A' of the directrix X', while the other extremity of the string is allowed to slip freely through the corresponding point A of a second directrix X and is kept

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* "How to draw a straight line," 1877.

† *Lecons de Cinematique*, 1897, pp. 302-307.

‡ *Transactions of the American Mathematical Society*, Vol. III (1902), pp. 493-498.

§ *Introduction to Projective Geometry and Its Applications*, 1905, pp. 242-260.

|| A regulus is a ruled quadric.

¶ *Kinematische Gelenksysteme und die durch sie erzeugten geometrischen Transformationen*, Solothurn, 1906. *Archives der sciences physiques et naturelles*, Vol. 24 (1907), p. 368.

taut by a weight at its end P, so that, no matter what the relative position of X' and X may be, AP is always in a vertical position.

If the set of all points A is projective to the corresponding set A', then, as is well known, the strings represent one of the systems of generatrices of a regulus, while X and X' are two elements of the other system. An infinite number of reguli are obtained by changing the relative position of X and X' in space.

Assuming a general projective relation between the points X and X', then the locus of all points P is a quartic.

To prove this proposition, X may be assumed through the origin and in the $\xi\zeta$-plane of a rectangular system of coördinates in space and X' as a line parallel to the ζ-axis and intersecting the η-axis at a distance p from the origin. This can be done without loss of generality. In Fig. 1, let A and A' be two corresponding points on X and X', and let $OA=x$ and $O'A'=x'$, then

$$x'=\frac{ax+b}{cx+d}.$$

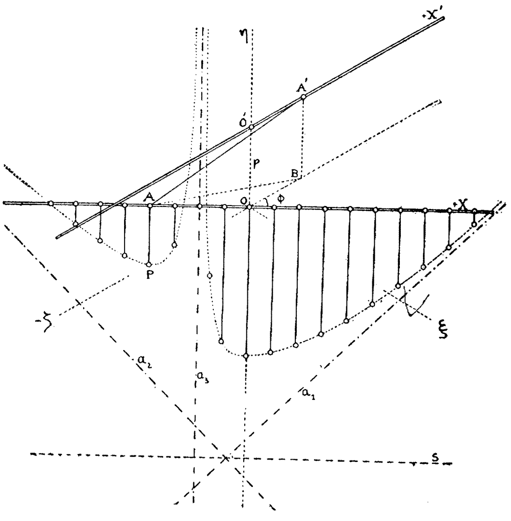

Fig. 1.

If B is the projection of A' on the $\xi\zeta$-plane, and ϕ the angle which the projections of X' upon the same plane includes with X, then from the figure, $(A'A)^2=(AB)^2+(A'B)^2$. But $(AB)^2=(OA)^2+(OB)^2-2OA.OA\cos\phi=x^2+\left(\frac{ax+b}{cx+d}\right)^2-2x\frac{ax+b}{cx+d}\cos\phi$, hence,

$$(A'A)^2=x^2+\left(\frac{ax+b}{cx+d}\right)^2-2x\frac{ax+b}{cx+d}\cos\phi+p^2.$$

Now $A'A+AP$ equals the constant length l of the strings, so that $AA'=l-AP$, or designating the ordinate of the required curve by $AP=y$,

$$(1)\quad (l-y)^2=x^2+\left(\frac{ax+b}{cx+d}\right)^2-2x\frac{ax+b}{cx+d}\cos\phi+p^2, \text{ or}$$

$$[x^2-(l-y)^2+p^2](cx+d)^2-2x(ax+b)(cx+d)\cos\phi+(ax+b)^2=0.$$

This is the equation of the required curve, which is of the fourth order and which depends on two variable parameters p and ϕ. By changing these arbitrarily a two parameter system of quartics is obtained. The quartics corresponding to all other positions of X and X' result from linear transformations of the first system.

It can easily be established that all curves of system (1) have the three lines a_1, a_2, a_3 with the equations

$$y=x+10-\frac{a}{c}\cos\phi, \quad y=-x+10+\frac{a}{c}\cos\phi, \quad x=-\frac{d}{c}$$

as common asymptotes. In Fig. 1, which represents an isometric projection of the regulus and a curve of the system, $a=1$, $b=-2$, $c=1$, $d=2$, $p=4$, $\phi=45°$. Only one complete string, $A'AP$, has been drawn, and the part of the curve which is mechanically possible is indicated by its ordinates. In order to obtain the complete curve as represented by the equation the part shown must be reflected on the line s.

3. In subways with circular cross sections and lined with glazed tiling one may observe peculiar curves of reflexion emenating from the incandescent lights and winding on the surface in an ultimate direction towards the eye of the observer.*

To study the nature of these curves, assume in Fig. 2, a perspective view, L as a source of light near the cylindrical surface of the subway and B as the eye of the observer.

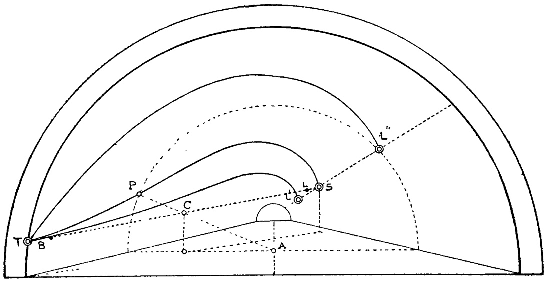

Fig. 2.

It is clear that every ray emanating from L that strikes the surface at P and is reflected to B is coplanar with the reflected ray PB. Furthermore, the perpendicular of incidence is a normal to the surface and consequently passes through the axis of the cylinder. Hence, in order to obtain points of reflexion P, pass a pencil of planes through the line connecting L and B. Each of these planes cuts the axis of the cylinder in a point A, and the surface normal from A, cutting LB at C, meets the surface in a point P of the curve containing reflecting points.

All normals PA from points of the curve pass through LB and the axis of the cylinder and are parallel to the planes normal to the axis; they form therefore an hyperbolic paraboloid. *The curve thus appears as the intersection of this regulus and the cylinder and is conseqnently of the fourth order.*

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* Seen in the "Traforo," a subway under the Royal Gardens of the Quirinale in Rome.

It passes through the piercing points S and T, near L and B, of LB with the cylindrical surface. Every point of the cylinder gives rise to such a curve, so that a doubly infinite number of curves is obtained which all pass through T. In Fig. 2, three lamps at L, L', L'' at equal intervals and their corresponding curves are shown.

A number of problems suggest themselves in connection with a detailed study of these curves. As an example the result may be stated, that the locus of the foci of all conics (ellipses) cut out on the surface by the planes of the pencil through LB is a twisted quartic, whose orthogonal projection on a plane normal to LB passes through the circular points.

Referring again to Fig. 2, it is clear that the quartic through S and T is determined by ST and does not depend on the position of L and B on ST. Hence assuming on ST, L and B arbitrarily, there are generally only a limited number of points on the quartic where reflexion towards B takes place, so that geometrically there are no continuous curves of reflexion.

The reason why such curves are seen physically lies in the fact that in reality the tunnel-surface consists of rectangular plane pieces and the source of light of a luminous body. In place of an incident and reflected ray at each point of the quartic we therefore have bundles of rays striking the plane portions around each point of the curve. On account of the proximity of L and B to the surface, parts of each bundle will be reflected to B.

ON COMPOSITE NUMBERS P WHICH SATISFY THE FERMAT CONGRUENCE $a^{P-1}\equiv 1 \bmod P$.*

By R. D. CARMICHAEL, Indiana University.

Professor J. H. Jeans† has discussed the question of the converse of Fermat's theorem, showing that the relation

$$a^{P-1}\equiv 1 \bmod P, \tag{1}$$

which (by Fermat's theorem) is always true when P is a prime for any value of a which is prime to P, is for any particular value of a true for values of P which are not prime. Mr. E. B. Escott‡ has given a more direct proof of the same theorem.

The failure of the converse of Fermat's theorem has also been pointed out by Lucas§ by means of the example

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[*] Read before the American Mathematical Society, October 28, 1911.

[†] *Messenger of Mathematics*, 27 (1897-8), p. 174.

[‡] *Messenger of Mathematics*, New Series, No. 431 (1907), p. 175.

[§] *Theorie des nombres*, I, p. 422.